Intro to Algebraic Geometry

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1 Introduction

Some basic questions we aim to consider are

- How do we know when 2 collections of polynomials have same vanishing set? *Nullstellensatz*
- What do varieties look like locally? *smoothness & singularities*
- How do we build new varieties from old ones? *morphisms & blowups*
- When are varieties "equivalent"? *moduli & birational geometry*
- How do curves intersect? *intersection theory*
- Relation of algebraic geometry with other fields

2 Algebraic Sets

Definition 2.1. The *n-dimensional affine space* $\mathbb{A}_k^n$ over k is k^n with vector space $k[X_1, \cdots, X_n]$.

Remark 2.2. Every $f \in k[X_1, \cdots, X_n]$ is an evaluation polynomial on $x \in \mathbb{A}_k^n$ where we will write $f(x)$ to denote the evaulation.

Definition 2.3. For $S \subseteq k[X_1, \cdots, X_n]$, its *vanishing set* is

$$V(S) = \left\{x \in \mathbb{A}_k^n \mid \forall P \in S, P(x) = 0\right\},$$

where $V(S)$ is an *algebraic set* aka *closed embedded affine variety*.

Example 2.4. If $n = 0$, then $V(1) = \emptyset$ and $V(0) = \mathbb{A}_k^n$.

Example 2.5. If $n = 2$, we have "lines" $V(y - ax - b) = \{(x, y) \in \mathbb{A}_k^2 \mid y = ax + b\}$ and "conics" $V(a_0x^2 + a_1xy + a_2y^2 + b_0x + b_1y + c)$.

Example 2.6. When $n = 2$, take the points $\left(\frac{p}{u}, \frac{q}{v}\right)$ in $V = V(x^2 + y^2 = 1)$ over $\mathbb{Q}$.

Claim 2.7. *There are infinitely many primitve Pythagorean triples.*

Proof. V forms the integers $u, v, p, q \in \mathbb{Z}$ such that $(vp)^2 + (uq)^2 = (uv)^2$. Therefore, we find them to be in 1-1 correspondence with Pythagoren triples

$$V \leftrightarrow \{\text{primitive Pythagoren triples}\}.$$

By the correspondence above, we only need to show that there are infinitely many points in V. Taking any point in $V \setminus \{(0,1)\}$ and finding the projection from $(0,1)$ onto the x-axis gives a bijection

$$V \setminus \{(0,1)\} \leftrightarrow \mathbb{A}^1_{\mathbb{Q}} = \mathbb{Q}.$$

□

Example 2.8. The vanishing set depends on the choice of k. Take $f = X^2 - c \in k[X]$.

- If $c = 0$, then $V(f) = \{0\}$ for any k.
- If $c \neq 0$, then $V(f) = \begin{cases} \{\pm\sqrt{c}\} & \text{if } c \text{ has square root in } k. \\ \emptyset & \text{else.} \end{cases}$

From now on, we assume our field k is algebraically closed and characteristic 0.

3 Regular Functions

Definition 3.1. For algebraic set $V \subseteq \mathbb{A}^n$, map $f \colon V \to k$ is a *regular function* if it is the restriction of a polynomial function on $\mathbb{A}^n$. Denote $\mathcal{O}(V)$ the set of regular functions on V.

Proposition 3.2. *$\mathcal{O}(V)$ is a k-algebra.*

Proof. $\mathcal{O}(V)$ is a unital ring under pointwise addition and multiplication by $(f+g)\big|_V = f\big|_v + g\big|_V$ and $(fg)\big|_V = f\big|_V g\big|_V$ with identities given by constant functions $0, 1$. We also have an injective ring map $k \hookrightarrow \mathcal{O}(V)$, so we have a k-algebra. □

Corollary 3.3. *Restriction defines a surjective algebra map $\mathcal{O}(\mathbb{A}^n) \twoheadrightarrow \mathcal{O}(V)$. From this map we find $\mathcal{O}(V) = \mathcal{O}(\mathbb{A}^n)/I$ for ideal $I = \{f \in \mathcal{O}(\mathbb{A}^n) \mid f\big|_V = 0\}$*

4 Hilbert's Nullstellensatz

Definition 4.1. The *radical of ideal I* in commutative ring R is

$$\sqrt{I} = \{r \in R \mid r^n \in I, n \in \mathbb{Z}^+\}.$$

An ideal is *radical* if $I = \sqrt{I}$.

Theorem 4.2 (Hilber's Basis Theorem)**.**

First, we define maps

$$\begin{tikzcd} \{T \subseteq \mathcal{O}(\mathbb{A}^n)\} \arrow[r, bend left, "V"] & \{W \subseteq \mathbb{A}^n\} \arrow[l, bend left, "I"] \end{tikzcd}$$

$$\begin{aligned} T \quad & \mapsto V(T) = \{p \in \mathbb{A}^n \mid f(p) = 0 \forall f \in T\} \\ I(W) = \{f \in \mathcal{O}(\mathbb{A}^n) \mid f(p) = 0 \forall p \in W\} \hookleftarrow & \quad W \end{aligned}$$

Remark 4.3. Immediately, we can see that V and I form a Galois connection since $W \subseteq W' \implies T(W) \subseteq T(W')$ and $T \subseteq T' \implies V(T) \supseteq V(T')$

Lemma 4.4. *For $W \subseteq \mathbb{A}^n$, we have $I(W) \trianglelefteq \mathcal{O}(\mathbb{A}^n)$ is radical.*

Proof. First, to show that $I(W) \trianglelefteq \mathcal{O}(\mathbb{A}^n)$, take any $w_1, w_2 \in I(W), t_1, t_2 \in \mathcal{O}(\mathbb{A}^n)$. Then,

$$\begin{aligned} w_1\big|_W = w_2\big|_W = 0 &\Rightarrow t_1 w_1\big|_W = t_2 w_2\big|_W = 0 \\ &\Rightarrow t_1 w_1\big|_W + t_2]rstw_2W = 0 \\ &\Rightarrow (t_1 w_1 + t_2 w_2)\big|_W = 0 \end{aligned}$$

which show $t_1 w_1 + t_2 w_2 \trianglelefteq \mathcal{O}(\mathbb{A}^n)$. To show $I(W)$ is radical, it's first evident that $\sqrt{I(W)} \subseteq I(W)$. Then, we have

$$\begin{aligned} w \in \sqrt{I(W)} &\implies w^i \in I(W) \text{ for } i \in \mathbb{Z}^+ \\ &\implies w(p)^i = 0 \forall p \in W \\ &\implies w(p) = 0 \\ &\implies w \in I(W) \end{aligned}$$

□

Lemma 4.5. *For $T \subseteq \mathcal{O}(\mathbb{A}^n), \exists t_1, \cdots, t_l \in \mathcal{O}(\mathbb{A}^n)$ s.t. $V(T) = V((T)) = V(t_1, \cdots, t_l)$.*

Proof. To see $V(T) = V((T))$, it's first evident that $V((T)) \subseteq V(T)$ by the Galois connection above. Then, take any $p \in V(T)$ and $t = \sum_{i=1}^{l} f_i t_i \in (T) : f_i \in \mathcal{O}(\mathbb{A}^n), t_i \in T$. Notice that $g_i(p) = 0 \forall i$, and so $p \in V((T))$. Now, we can assume that T is an ideal. By Hilbert's Basis Theorem 4.2, $\mathcal{O}(\mathbb{A}^n) = k[X_1, \cdots, X_n]$ is Noetherian, and so T is f.g. as $T = (t_1, \cdots, t_l)$ for some $t_1, \cdots, t_l \in T$. □

By Lemma 4.4 and Lemma 4.5, we can restrict the above maps to

$$\begin{tikzcd} \{\text{radical } T \trianglelefteq \mathcal{O}(\mathbb{A}^n)\} \arrow[r, bend left=60, "V"] & \{\text{algebraic } W \subseteq \mathbb{A}^n\} \arrow[l, bend left=60, "I"] \end{tikzcd}$$

Theorem 4.6 (Hilbert's Nullstellensatz)**.** *The maps V, I are mutual inverses:*

1. *For algebraic set $W \subseteq \mathbb{A}^n, V(I(W)) = W$.*
2. *For $T \trianglelefteq \mathcal{O}(\mathbb{A}^n), I(V(T)) = \sqrt{T} = T$ if T radical.*

5 Zariski Topology

Definition 5.1. For set S, a *topology* on S is a family of subsets called the closed subsets such that

1. $\emptyset$ and S are closed.
2. if $X_1, \cdots, X_i$ are closed then so is $X_1 \cup \cdots \cup X_i$.
3. if $X_i : i \in \mathcal{I}$ are closed for arbitrary index set $\mathcal{I}$ then so is $\bigcap_{i \in \mathcal{I}} X_i$.

A subset of S is *open* if its complement is closed. A *topological space* is a set with a topology.

Definition 5.2. Subset T of topological space S is *dense* if any of the following equal conditions hold.

1. $\operatorname{cl} T = S$.
2. $\forall s \in S$, every neighborhood U of s intersects S.

Theorem 5.3. *The algebraic sets form a topology on $\mathbb{A}^n$.*

Proof. 1. $V(\mathcal{O}(\mathbb{A}^n)) = \emptyset$ and $V(\{0\})$ are algebraic.

2. Take closed $V_1 = V(I_1), V_2 = V(I_2) \subseteq \mathbb{A}^n$. We claim that $V_1 \cup V_2 = V(I_1 \cap I_2) = V(I_1 I_2)$. First, notice that

$$I_1 I_2 \subseteq I_1 \cap I_2 \subseteq I_1 \cup I_2 \implies V(I_1 I_2) \supseteq V(I_1) \cap V(I_2) \supseteq V_1 \cup V_2$$

by the Galois connection. Conversely, take $p \in V(I_1 I_2)$. If $p \in V_1$, then we are done so suppose $p \notin V_1$. Then, let $f_1 \in I_1$ be such that $f_1(p) \neq 0$. For any $f_2 \in I_2$, we have since $p \in V(I_1 I_2)$ that $f_1(p) f_2(p) = 0 \implies f_2(p) = 0$, and so $p \in V(I_2) \forall f \in I_1 I_2, f(p) = 0$. By induction, we find closure for any finite union.

3.

$$\begin{aligned}
\bigcap_{i \in \mathcal{I}} V_i &= \bigcap_{i \in \mathcal{I}} V(I_i) \\
&= \{p \in \mathbb{A}^n \mid f(p) = 0 \forall f \in I_i \forall i \in \mathcal{I}\} \\
&= V\left(\bigcup_{i \in \mathcal{I}} V(I_i)\right)
\end{aligned}$$

□

Definition 5.4. The topology of algebraic sets in Theorem 5.3 on $\mathbb{A}^n$ is the *Zariski topology*.

Remark 5.5. More generally the algebraic subsets $V' \subseteq V$ for algebraic set $V \subseteq \mathbb{A}^n$ form a topology on V, which is the subspace topology on V and called the Zariski topology of V.

Lemma 5.6. *$V \subsetneq \mathbb{A}^1$ is closed iff it is finite.*

Definition 5.7. For algebraic set V and $W \subseteq V$, the *closure* of W is the smallest closed subset of V containing W

$$\overline{W} = \bigcap_{\substack{W \subseteq U \subseteq V \\ \text{algebraic}}} U = V(I(W))$$

6 Irreducibility

Definition 6.1. Algebraic set V is *reducible* if $\exists$ closed $V_1, V_2 \subsetneq X$ s.t. $V = V_1 \cup V_2$. Else, V is *irreducible*.

Example 6.2. Let $I(V) = (f)$ for $f \in \mathcal{O}(\mathbb{A}^n)$. Then V is irreducible iff f is irreducible.

Proposition 6.3. *For algebraic set $V \subseteq \mathbb{A}^n$, TFAE*

1. *V is irreducible.*
2. *$\forall$ open, nonempty $U, V \subseteq V, U \cap V \neq \emptyset$.*
3. *Any open, nonempty $U \subseteq V$ is dense.*
4. *Ideal $I(V) \subseteq \mathcal{O}(\mathbb{A}^n)$ is prime.*
5. *Algebra $\mathcal{O}(V) \cong \mathcal{O}(\mathbb{A}^n)/I(V)$ is a domain.*

Example 6.4. $\mathbb{A}^n$ is irreducible since $k[X_1, \cdots, X_n]$ is a domain.

Theorem 6.5. *Every algebraic set $V \subseteq \mathbb{A}^n$ can be written as $V_1 \cup \cdots \cup V_n$ for closed, irreducible $V_1, \cdots, V_n \subseteq V$ and $V_i \nsubseteq X_j$ if $i \neq j$, unique up to permutation.*

Definition 6.6. As in Theorem 6.5, $V = V_1 \cup \cdots \cup V_n$ is its *irreducible decomposition* where $V_1, \cdots, V_n$ are the *irreducible components*.

7 Regular Maps

Definition 7.1. For algebraic sets $V_1 \subseteq \mathbb{A}^n, V_2 \subseteq \mathbb{A}^m$, map $\varphi : V_1 \to V_2$ is a *regular map* if $\varphi = (\varphi_1, \cdots, \varphi_m)$ where $\varphi_i \in \mathcal{O}(V_1)$ such that $\operatorname{im}(\varphi_1, \cdots, \varphi_m) \subseteq V_2$.

Remark 7.2. The algebraic sets with regular maps as morphisms form a category. The regular maps $\varphi : V_1 \to V_2$ correspond contravariantly bijectively with k-algebra homomorphisms $\varphi^* : \mathcal{O}(V_2) \to \mathcal{O}(V_1), \psi \mapsto \psi\varphi$ via post-composition.

Lemma 7.3. *Every regular map $\varphi : V_1 \to V_2$ is continuous in the Zariski topology.*

Definition 7.4. A regular map $\varphi : V_1 \to V_2$ is an *isomorphism* if it is bijective and φ^{-1} is regular.

8 Dimension

Definition 8.1. The *dimension* of algebraic set $V \subseteq \mathbb{A}^n$ is

$$\dim V = \sup\{m \in \mathbb{Z}_{\geq 0} \mid \exists V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_m : V_i \text{ closed, irreducible } \forall i\}.$$

Equivalenty, $\dim V$ is the *Krull dimension* of $\mathcal{O}(V)$ given by

$$\dim \mathcal{O}(V) = \sup\{m \in \mathbb{Z}_{\geq 0} \mid \exists \mathcal{O}(V) \supsetneq I_0 \supsetneq I_1 \supsetneq \cdots \supsetneq I_m : I_i \text{ prime } \forall i\}.$$

Remark 8.2. Immediately, we have $V_1 \subseteq V_2 \implies \dim V_1 \subseteq \dim V_2$.

Theorem 8.3. $\dim \mathbb{A}^n = n$.

Corollary 8.4. *If* $V \subsetneq \mathbb{A}^n$*, then* $\dim V < n$.

Definition 8.5. If algebraic sets $V_1 \subseteq V_2$ are, then the *codimension* of V_1 in V_2 is

$$\operatorname{codim}(V_1, V_2) = \dim(V_2) - \dim(V_1)$$

Theorem 8.6 (Special Case of Krull's Hauptidealsatz)**.** *For algebraic set* $V \subseteq \mathbb{A}^n$*, TFAE*

1. *Every irreducible component of* V *has dimension* $n - 1$.
2. $V = V(f)$ *for some* $f \neq 0 \in \mathcal{O}(\mathbb{A}^n)$.

Definition 8.7. A *hypersurface* is an algebraic set V where $\dim V = n - 1$.

9 Tangent Spaces

The *Zariski tangent space* of algebraic set $V \subseteq \mathbb{A}^n$ at point $p \in V$ is

$$T_p X = \{v \in \mathbb{A}^n \mid \nabla f\big|_p \cdot v = 0 \forall f \in I(X)\}$$

Proposition 9.1. *If* $I(V)$ *is generated by* $f_1, \cdots, f_j$*, then for* $p \in V, v \in T_p V \iff v \in \ker J(f_1, \cdots, f_j)$*, where* J *is the Jacobian.*

Proof. Let $f = g_1 f_1 + \cdots + g_n f_n$ for $g_i \in V$. Then,

$$\begin{aligned} \nabla f\big|_p &= \sum_{i=1}^n g_i(p) \nabla f_i\big|_p + \nabla g_i\big|_p f_i \\ &= \sum_{i=1}^n g_i(p) \nabla f_i\big|_p \end{aligned}$$

since $f_i(p) = 0$. Therefore, $v \in T_p V \iff \nabla f_i\big|_p \dot{v} = 0 \iff v \in \ker(J(f_1, \cdots, f_j))$. □

Example 9.2. Take $V = V(xy)$ where $I(V) = (xy)$ generated by $f = xy$. Then,

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} y & x \end{pmatrix}$$
$$= \begin{cases} \begin{pmatrix} 0 & x \end{pmatrix} & x-\text{axis} \\ \begin{pmatrix} y & 0 \end{pmatrix} & y-\text{axis} \\ \begin{pmatrix} 0 & 0 \end{pmatrix} & \text{origin} \end{cases}$$

Via Prop 9.1, we find the tangent space

$$T_pV = \ker J = \begin{cases} \left\{ \begin{pmatrix} * \\ 0 \end{pmatrix} \middle| * \in k \right\} & p \text{ on } x-\text{axis} \setminus \{(0,0)\} \\ \left\{ \begin{pmatrix} 0 \\ * \end{pmatrix} \middle| * \in k \right\} & p \text{ on } x-\text{axis} \setminus \{(0,0)\} \\ k^2 & p \text{ at } (0,0) \end{cases}$$

Example 9.3. If $V \subseteq \mathbb{A}^n$ is a hypersurface with $I(V) = (f)$, then

$$J(f) = \begin{pmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{pmatrix}$$

and so there are 2 possibilities:

$$J(f)\big|_p \neq 0 \implies \dim T_pV = n - 1$$
$$J(f)\big|_p = 0 \implies T_pV = k^n$$

T_pV as Derivations

Definition 9.4. For k-algebra A with $a, b \in A$, a *derivation* is a k-linear map $D : A \to A$ that satisfies the *Leibniz rule*:

$$D(ab) = D(a)b + aD(b).$$

Definition 9.5. For point $p \in \mathbb{A}^n$, direction vector $v \in k^n$, and $f \in \mathcal{O}(\mathbb{A}^n)$, the *directional derivative* is

$$v_p(f) = \nabla f\big|_p \cdot v$$

which gives a k-linear map $\mathcal{O}(\mathbb{A}^n) \to k$, $f \mapsto v_p(f)$.

Lemma 9.6. *If D is a derivation, then $D(\lambda) = 0 \forall \lambda \in k$.*

Proof. First, note that $D(1) = D(1 \cdot 1) = 1D(1) + D(1)1 \implies D(1) = 0$. Then, by linearity, $D(\lambda) = \lambda D(1) = 0$. □

Lemma 9.7. *Every derivation D of $\mathcal{O}(\mathbb{A}^n)$ based at point $p \in \mathbb{A}^n$ is a directional derivative.*

Proof. By Taylor expansion around point p and linearity, we have

$$\begin{aligned}
D(f) &= \sum_{i=0}^{\infty} \frac{1}{i!} D\left(\nabla^i f\big|_p \cdot (x-p)^i\right) \\
&= D(f(p)) + \nabla f\big|_p \cdot \underbrace{D(x-p)}_{v} + \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}\bigg|_p (D((x_i-p_i)(x_j-p_j))) + \cdots \\
&= 0 + v_p(f) + \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}\bigg|_p (D(x_i-p_i) \underbrace{(x_j-p_j)}_{p_j-p_j=0} + \underbrace{(x_i-p_i)}_{p_i-p_i=0} D(x_j-p_j)) + \cdots \\
&= v_p(f)
\end{aligned}$$

□

Corollary 9.8. *$T_p\mathbb{A}^n \cong \{$derivations $\mathcal{O}(\mathbb{A}^n) \to k$ based at $p\}$ and has a canonical basis* $\frac{\partial}{\partial x_1}\Big|_p, \cdots, \frac{\partial}{\partial x_n}\Big|_p$

Corollary 9.9. *$T_pV^n \cong \{$derivations $\mathcal{O}(V) \to k$ based at $p\}$.*

T_pV as Ring Homomorphisms

Definition 9.10. Ring $k[\varepsilon]/\varepsilon^2 \cong k \oplus k\varepsilon$ is the *dual numbers*

Proposition 9.11. *There is an isomorphism between derivations based at p and certain homomorphisms of $\mathcal{O}(V) \to k[\varepsilon]/\varepsilon^2$.*

Proof. For derivation $v : \mathcal{O}(V) \to k$ at p, map $\widetilde{v} : \mathcal{O}(V) \to k[\varepsilon]/\varepsilon^2, f \mapsto f(p) + v(f)\varepsilon$ is a ring homomorphism. We can check

$$\begin{aligned}
\widetilde{v}(f)\widetilde{v}(g) &= (f(p) + v(f)\varepsilon)(g(p) + v(g)\varepsilon) \\
&= f(p)g(p) + \underbrace{v(f)g(p)\varepsilon + f(p)v(g)\varepsilon}_{v \text{ is derivation}} + \underbrace{v(f)v(g)\varepsilon^2}_{0} \\
&= (fg)(p) + v(fg)\varepsilon \\
&= \widetilde{v}(fg)
\end{aligned}$$

□

T_pV as Thickened Points

T_pV as Quotient of Maximal Ideals

Definition 9.12. The dual vector space of T_pV is the *cotangent space* and denoted $T_p^*V \cong m_p/m_p^2$

10 Smoothness and Singularities

Definition 10.1. The *local dimension* of algebraic set $V \subseteq \mathbb{A}^n$ at $p \in V$ is

$$\dim_p V = \max\{\dim W \mid W \text{ irreducible component of } V : p \in W\}$$

Definition 10.2. Point $p \in V$ is smooth if $\dim T_pV = \dim_p V$. Else, p is singular and $\dim T_pV > \dim_p V$. The *singular locus* and *regular locus* of V are the sets of singular and smooth points i.e.

$$\begin{aligned} V_{\text{sing}} &= \{p \in V \mid \dim T_pV \neq \dim_p V\} \\ V_{\text{reg}} &= \{p \in V \mid \dim T_pV = \dim_p V.\} \end{aligned}$$

V is *smooth* or *nonsingular* if $V = V_{reg}$.

Theorem 10.3. *The following are true.*

1) $\forall j \geq 0$, $S_j(V) = \{p \in V \mid \dim T_pV \geq j\} \subseteq V$ *is closed.*

2) $\dim T_pV \geq \dim_p V \forall p \in V$.

3) If $p \in V$ is contained in more than one irreducible component of V, then p is singular. Else, p is in a unique irreducible component $V_p \subseteq V$, and p is smooth in V iff p is smooth in V_p.

4) V_{reg} is open and dense. Equivalenty, V_{sing} is closed and does not contain an irreducible component of V.

Definition 10.4. For commutative ring R with ideal $I \trianglelefteq R$, the *completion of R along I* is $\widehat{R}_I = \{(f_1, f_2, \cdots,) \in \prod_l R/I^l \mid f_i = f_j \mod I^i \forall i \leq j\}$.

Remark 10.5. Equivalenty, the competion is given as the inverse limit

$$R/I \twoheadleftarrow R/I^2 \twoheadleftarrow R/I^3 \twoheadleftarrow \cdots$$

where we have $\widehat{R}_I = \lim_l R/I^l$.

Remark 10.6. Let $\mathfrak{m} = (x_i)_{i \in [n]} \trianglelefteq \mathcal{O}(\mathbb{A}^n)$ be the maximal ideal corresponding to 0. Then, quotients correspond to truncating powers as $\mathcal{O}(\mathbb{A}^n)/\mathfrak{m}^l \cong \{\text{polynomials with deg} < l\}$, which gives the commutative diagram

$$\begin{tikzcd}
& & \vdots \arrow[d, two heads] \\
& & \mathcal{O}(\mathbb{A}^n)/\mathfrak{m}^l \arrow[d, two heads] \\
\mathcal{O}(\mathbb{A}^n) \arrow[r, hook] \arrow[rru, hook, bend left] \arrow[rr, hook, bend left] \arrow[rrd, hook, bend right] & k[\![x_1, \cdots, x_n]\!] \arrow[ru, two heads] \arrow[r, two heads] \arrow[rd, two heads] & \vdots \arrow[d, two heads] \\
& & \mathcal{O}(\mathbb{A}^n)/\mathfrak{m} \cong k
\end{tikzcd}$$

Definition 10.7. For algebraic set V and $p \in V$, the *completion of $\mathcal{O}(V)$ along maximal ideal* $\mathfrak{m}_p$ is

$$\widehat{\mathcal{O}}_{V,p} = \lim_{l} \mathcal{O}(V)/\mathfrak{m}_p^l$$

Remark 10.8. We have map $\widehat{\mathcal{O}}_V, p \twoheadrightarrow k, (f_1, f_2, \cdots) \to f_1(p) \quad (= f_j(p) \forall j)$ with kernel a maximal ideal $\widehat{\mathfrak{m}}_p \subseteq \widehat{\mathcal{O}}_V, p$.

Proposition 10.9. *The following are true.*

1) The canonical homomorphism $\mathcal{O}(V) \hookrightarrow \widehat{\mathcal{O}}_{V,p}$ is injective. (Follows from Nakayam's Lemma)

2) $\widehat{\mathcal{O}}_{V,p}$ has no nonzero nilpotent elements.

3) $\widehat{\mathcal{O}}_{V,p}$ is Noetherian.

4) $\widehat{\mathcal{O}}_{V,p}/\widehat{\mathfrak{m}}_p^l \cong \mathcal{O}(V)/\mathfrak{m}_p^l \forall l \geq 0$

5) $\widehat{\mathfrak{m}}_p/\widehat{\mathfrak{m}}_p^2 \cong \mathfrak{m}_p/\mathfrak{m}_p^2$

Theorem 10.10. *Let $n = dim T_p V$ and suppose $x_1, \cdots, x_n \in \widehat{\mathfrak{m}}_p$ project to a basis for $T_p^* V \cong \widehat{\mathfrak{m}}_p/\widehat{\mathfrak{m}}_p^2$. Then, $\exists f_1, \cdots, f_j \in \widehat{\mathfrak{m}}_p^2$ s.t.*

$$\widehat{\mathcal{O}}_{V,p} \cong k[\![x_1, \cdots, x_n]\!]/(f_1, \cdots, f_j)$$

Corollary 10.11. *Let $n = \dim T_p V$ and supposed $p \in V$ is a smooth point. Then,*

$$\widehat{\mathcal{O}}_{V,p} \cong k[\![x_1, \cdots, x_n]\!] \cong \widehat{\mathcal{O}}_{\mathbb{A}^n,0}$$

Example 10.12. Take $V = V(y^2 - x^3 - x^2)$ where $I(V) = (y^2 - x^3 - x^2)$ generated by $f = y^2 - x^3 - x^2$. Then,

$$\frac{\partial f}{\partial x} = -x(3x+2)$$
$$\frac{\partial f}{\partial y} = 2y$$

Therefore, we find

$$\begin{aligned} V_{\text{sing}} &= V(y^2 - x^3 - x^2, x(3x+2), 2y) \\ &= V(x^2(x+1), x(3x+2), y) \\ &= \{0\} \cup V(\underbrace{x+1, 3x+2}_{\text{no common zeros}}, y) \\ &= \{0\} \end{aligned}$$

11 Cones

Definition 11.1. A *cone* in $\mathbb{A}^n$ with vertex $p \in \mathbb{A}^n$ is an algebraic set $W \subseteq \mathbb{A}^n$ s.t. $q \in W \implies \overline{pq} \subseteq W$ or $W = \bigcup_{\text{lines } L \subseteq W}$

Example 11.2. Examples of cones with vertex at 0: $V(xy), V(xy(x-y)(x-\lambda y)) \subseteq \mathbb{A}^2$ and $V(x^2 + y^2 - z^2) \subseteq \mathbb{A}^3$.
Nonexamples: cusp $V(y^2 - x^3) \subseteq \mathbb{A}^2$, Whitney umbrella $V(x^2 - zy^2)$

Definition 11.3. Polynomial $f \in \mathcal{O}(\mathbb{A}^n) = k[x_1, \cdots, x_n]$ is *homogenous of degree d* if it is a linear combination of monomials each with total degree of d i.e.

$$f = \sum_{i_1 + \cdots + i_n = d} a_{i_1, \cdots, i_n} x_1^{i_1} \cdots x_n^{i_n} \in k[x_1, \cdots, x_n]$$

Proposition 11.4. *f is homogenous of degree d iff*

$$f(\lambda p) = \lambda^d f(p) \forall \lambda \in k, p \in k^n$$

Proof. We prove both directions.
$(\implies)$: Plugging in λ gives it immediately by linearity.
$(\impliedby)$: If f were not homogenous of degree d, then $\exists p \in k^n, \lambda \neq 0 \in k$ s.t. $f(p) \neq 0$ and $f(\lambda p) = 0$. To see this, note $f \neq$, so pick $p \in k^n$ s.t. $f(p) \neq 0$. Observe $f(\lambda p)$ must vanish at a nonzero λ by dividing out until largest exponent of λ. Then, $f(\lambda p) = \lambda^d f(p) \forall \lambda \in k, p \in k^n$, but $f(\lambda p) = 0$ so $\lambda = 0$ which is a contradiction.

□

Lemma 11.5. *For closed $X \subseteq \mathbb{A}^n$, TFAE*

1) *X is cone with vertex* 0.
2) *X is closed under rescaling i.e. $p \in X \implies \lambda p \in X \forall \lambda \in k$.*
3) *$I(X) = (f_1, \cdots, f_l)$ for homogenous polynomials $f_1, \cdots, f_l$.*

Definition 11.6. For algebraic set V s.t. $0 \in V$, the *initial term* of polynomial $f \in I(V)$, is the homogeneous component of lowest degree in f denoted $\operatorname{in}(f)$. Correspondingly, the *initial ideal* is the ideal generated by the initial terms $\operatorname{in}(I)(V) = (\operatorname{in}(f))$. TODO: FIX UP

Definition 11.7. The tangent cone of V at 0 is

$$TC_0(V) = V(\operatorname{in}(I)(X)) \subseteq \mathbb{A}^n$$

Example 11.8. Let $V = V(y^2 - x^3)$, where $f = y^2 - x^3$. Then, $\operatorname{in}(f) = y^2$, and so $TC_0(V) = V(y^2)$ or the x-axis.

12 Projective Space

Definition 12.1. The *projective space* $\mathbb{P}^n$ is the set of all lines in $\mathbb{A}^{n+1}$ through 0. The *projectivization* of a finite dimensional k-vector space is $\mathbb{P}(V) = \{L \leq V \mid \dim L = 1\}$, so $\mathbb{P}^n = \mathbb{P}(k^{n+1})$.

Remark 12.2. We can define an equivalence relation $\sim$ by $v \sim w$ if $v = \lambda w$ for some $\lambda \in k \setminus \{0\}$. Then,

$$\mathbb{P}(V) \cong (V \setminus \{0\})/\sim .$$

Denote the equivalence class of $(x_0, \cdots, x_n) \in \mathbb{A}^{n+1} \setminus \{0\}$ by $[x_0 : \cdots : x_n]$.

Proposition 12.3. *$\mathbb{P}^n \cong \mathbb{A}^n \sqcup \mathbb{P}^{n-1}$ as sets.*

Proof. Take some $j \in [n]_0$ and let

$$U_j = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n \mid x_j \neq 0\}$$

Then, we have bijective map φ_j

$$\begin{tikzcd} U_j \arrow[r, bend left, "\varphi_j"] & \mathbb{A}^n \arrow[l, bend left, "\varphi^{-1}"] \end{tikzcd}$$

$$[x_0 : \cdots : x_n] \underset{\varphi^{-1}}{\overset{\varphi}{\longleftrightarrow}} (x_0/x_j, \cdots, x_{j-1}, x_j/x_j = 1, x_{j+1}, \cdots x_n/x_j)$$

The complement of U_j is $\mathbb{P}^n \setminus U_j = \{[x_1 : \cdots : x_n] \mid x_j = 0\}$, so we have bijection

$$\mathbb{P}^n \setminus U_j \cong \mathbb{P}^{n-1}$$
$$[x_1 : \cdots : x_{j-1} : 0 : x_{j+1} : \cdots : x_n] \longleftrightarrow (x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n)$$

□

Corollary 12.4. *By induction, $\mathbb{P}^n \cong \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \cdots \sqcup \mathbb{A}^1 \sqcup \mathbb{A}^0$.*

Corollary 12.5. *Each $[x_0 : \cdots : x_n] \in \mathbb{P}^n$ is s.t. $(x_0, \cdots, x_n) \neq 0$, so $\mathbb{P}^n = \bigcup_{j \in [n]_0} U_j$.*

Proposition 12.6. *For $X \subseteq \mathbb{P}^n$, TFAE*

1) *$\phi_j(X \cap U_j) \subseteq \mathbb{A}^n$ is closed $\forall j \in [n]_0$*

2) *Preimage $\pi^{-1(V)}$ of projection $\pi : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is closed in $\mathbb{A}^{n+1} \setminus \{0\}$*

3) *$\pi^{-1}(V) \cup \{0\} \subseteq \mathbb{A}^{n+1}$ is a cone with vertex 0.*

4) *V is the vainishing set of a collection of homogenous polynomials $F_1, \cdots, F_l \in \mathcal{O}(\mathbb{A}^{n+1})$.*

Definition 12.7. Any subset $V \subseteq \mathbb{P}^n$ that satisfies Prop 12.6 is called a *projective algebraic set*.

Corollary 12.8. *The closed sets of the quotient topoology on $\mathbb{P}^n \cong (\mathbb{A}^{n+1} \setminus \{0\})/k^*$ are the projective algebraic sets.*

Corollary 12.9. *$U_0 \cdots U_n \subseteq \mathbb{P}^n$ are open, so they form an open ccover of $\mathbb{P}^n$.*

Corollary 12.10 (Projective Nullstellensatz)**.**

$$\{closed\ T \in \mathbb{P}^n\} \cong \{radical\ W \trianglelefteq \mathcal{O}(\mathbb{A}^{n+1})\ generated\ by\ homogenous\ poly.\}$$

$$T \mapsto I(T)$$
$$V(W) \leftarrow\!\shortmid W$$

Corollary 12.11. *$V \subseteq \mathbb{P}^n$ is irreducible iff $I(V)$ is prime.*

Corollary 12.12. *The Zariski topology on $\mathbb{P}^n$ is Noetherian.*

Corollary 12.13. *Every closed $V \subseteq \mathbb{P}^n$ has a decomposition into a union of irreducible closed subsets, unique up to permutation.*

Definition 12.14. If $W \leq V$, then $\mathbb{P}(W) \leq \mathbb{P}(V)$ is called a *linear subspace*. It has *dimension* $dim_k W - 1$. If it has dimension d, then it is a *d-plane*. If it has codimension 1, then it is a *hyperplane*.

Remark 12.15. Any linear subspace $W \leq V$ is a cone and hence $\mathbb{P}(W) \leq \mathbb{P}(V)$ is closed.

Definition 12.16. For $X \subseteq \mathbb{P}(V)$, the *span of X* is the smallest linear subspace containing X i.e.

$$\operatorname{span} X = \bigcap_{\mathbb{P}(W) : X \subseteq \mathbb{P}(W)} \mathbb{P}(W) \leq \mathbb{P}(V)$$

Definition 12.17. Points $p_0, \cdots, p_l \in \mathbb{P}(V)$ are *linearly independent* if $\operatorname{span}\{p_0, \cdots, p_l\}$ is an l-plane. Else, they are *linearly dependent*.

Definition 12.18. A *homography* or *projective linear transformation* is an isormorphism of projective spaces derived from a linear isomorphism i.e. $\varphi : V \xrightarrow{\cong} W$ gives $\mathbb{P}(\varphi) : \mathbb{P}(V) \xrightarrow{\cong} \mathbb{P}(W)$

13 Location, Sheaves, & Varieties

Definition 13.1. For algebraic set $V \subseteq \mathbb{A}^n$ and open $U \subseteq V$, function $f : U \to k$ is regular if $\forall p \in U, \exists$ open subneighborhood $W \subseteq U$ of p and regular functions $g, h \in \mathcal{O}(V)$ s.t.

1. $V(h) \cap W \neq \emptyset$
2. $f\big|_W = \frac{g\big|_W}{h\big|_W}$

Proposition 13.2. *For algebraic set V with $f \in \mathcal{O}(V)$ and let $U_f = V \setminus V(f)$ be the associated open set. Then,*

$$\begin{aligned} \mathcal{O}(U_f) &= \left\{ \frac{g}{f^n} \mid g \in \mathcal{O}(V), n \geq 0 \right\} \\ &= \mathcal{O}(V)[f^{-1}] \\ &= \mathcal{O}(V)[t]/(tf - 1) \end{aligned}$$

which is also known as the "localization of $\mathcal{O}(V)$ at f."

Definition 13.3. For topological space X, a *presheaf* of k-algebras (respectively sets, groups, rings, $\cdots$) is an assignment

$$U \mapsto \mathcal{F}(U)$$

of a k-algebra (respectively set, group, ring, $\cdots$) to each open set $U \subseteq X$ and a k-algebra (respectively set, group, ring, $\cdots$) homomorphism

$$\mathcal{F}(U) \to \mathcal{F}(V)$$
$$s \mapsto s_{U,V} \text{ denoted } s\big|_V$$

for each open inclusion $V \subseteq U$ s.t.

1. $(-)_{U,U} = \mathrm{id}_{\mathcal{F}(U)}\ \forall$ open $U \subseteq X$

2. $(-)_{U,V}(-)_{V,W} = (-)_{U,W} \forall$ open $W \subseteq V \subseteq U$

A presheaf is a *sheaf* if the following 2 conditions hold $\forall$ open covers $\bigcup_{i\in I} U_i$

1. Locality: if $s, t \in \mathcal{F}(U)$ are s.t. $s\big|_{U_i} = t\big|_{U_i} \forall i \in I$, then $s = t$.
2. Glueing: if $s_i \in \mathcal{F}(U_i) \forall i \in I$ are s.t. $s_i\big|_{U_i\cap U_j} = s_j\big|_{U_i\cap U_j} \forall i, j \in I$, then $\exists s \in \mathcal{F}(U)$ s.t. $s\big|_{U_i} = s_i \forall i \in I$.

A *subsheaf* $\mathcal{G} \subseteq \mathcal{F}$ is a sheaf s.t. $\mathcal{G}(U) \subseteq \mathcal{F}(U) \forall$ open $U \subseteq X$.

Example 13.4. For topological space X,

$$\mathcal{F}\mathrm{un}_X^k : U \mapsto \{k\text{-valued functions on } U\}$$

forms a sheaf.

$$C_X^0 : U \mapsto \{\text{continuous } \mathbb{R}\text{-valued functions on } U\}$$

forms a subsheaf of $\mathcal{F}\mathrm{un}_X^{\mathbb{R}}$. If $V \subseteq \mathbb{A}^n$ is closed,

$$O_X : U \mapsto O(U)$$

is a subsheaf of $\mathcal{F}\mathrm{un}_x^k$.

Definition 13.5. A *(concrete) ringed space over k* is a topological space X equipped with a subsheaf $O_X \subseteq \mathcal{F}\mathrm{un}_X^k$.

Definition 13.6. An *algebraic variety over k* is a concrete ringed space $(V, O(V))$ if V is Noetherian and locally isomorphic to a closed subset of affine space.

Definition 13.7. An algebraic variety $(V, O(V))$ is

- *affine* if it is isomorphic to a closed subset of $\mathbb{A}^n$ for an $n > 0$.
- *quasi-affine* if it is isomorphic to an open subset of an affine variety (i.e. $V \cong W \setminus X$ for $X \subseteq W \subseteq \mathbb{A}^n$ closed).
- *projective* if it is isomorphic to a closed subset of a projective space.
- *quasi-projective* if it is isomorphic to an open subset of an open subset in a projective variety (i.e. $V \cong W \setminus Z$ for $Z \subseteq W \subseteq \mathbb{P}^n$ closed).

Remark 13.8. affine $\implies$ quasi-affine $\implies$ quasi-projective $\impliedby$ projective